# How to Use Matrices to Solve Systems of Linear Equations

## Or row reduction in 60 minutes

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# Outline

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# Outline

# Audience

This lesson is for you if you already know how to solve

$$\begin{aligned} x + y &= 9 \\ 2x - y &= 0 \end{aligned} \qquad \text{(two variables } x \text{ and } y\text{),}$$

and you want to learn a method for solving

$$\begin{aligned} 3x + 6y + 12z &= 9 \\ 2x + 4y - 4z &= 6 \\ y + 4z &= 7 \end{aligned} \qquad \text{(three variables } x,\ y, \text{ and } z\text{)}$$

or systems of linear equations with more than three variables. The method we'll cover can be applied to various kinds of problems, but we'll stick to systems of linear equations.

# What's in a Name?

## Gauss-Jordan Elimination

Two methods for solving systems of linear equations have similar names. In both methods, we can get the solution from a *matrix*, like the one below,

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

but each method uses a different type of matrix.

### Two Methods of Solution

| Method | Type of Matrix |
|---|---|
| Gaussian elimination | row echelon form |
| Gauss-Jordan elimination | *reduced* row echelon form |

We will be learning about Gauss-Jordan elimination, which builds off of Gaussian elimination.

# Overview

We will solve systems of linear equations in three stages. Before we get into details, the basic gist is as follows.

1. Convert system of equations to *augmented matrix*, a convienent way of representing the system without having to write $+$, $=$, $x$, $y$, or $z$!

$$\begin{aligned} 3x + 6y + 12z &= 9 \\ 2x + 4y - 4z &= 6 \\ y + 4z &= 7 \end{aligned} \qquad \longrightarrow \qquad \left[\begin{array}{ccc|c} 3 & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

2. Change rows of matrix, using allowable "operations", until matrix is "reduced."

3. Convert reduced matrix to new system of equations which is *much* easier to solve and has same solution (or solutions) as original system.

# Rows Versus Columns

In matrices (the plural form of the word matrix)...

- *Rows* are horizontal.
- *Columns* are vertical.

## Rows and Columns in a Matrix

$$\left[\begin{array}{ccc|c} 3 & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 3 & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

Row Column

## Why the Vertical Line?

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Row Column

## Why the Vertical Line?

It's a reminder that the matrix represents a system of equations: the line shows where the $=$ would go.

# Outline

# Which Operations Are Allowable?



The operations we're allowed to use when putting a matrix in reduced form (*reducing* it), are called ...

## The *Elementary Row Operations*

- Multiply a row by a nonzero constant.
- Add a multiple of one row to another.
- Interchange two rows (the rows need not be adjacent to each other).

Before discussing *why* each operation should be used, we will first consider examples of *how* each operation is used.

# Multiplying a Row by a Nonzero Constant



We can pick any row and multiply its entries by any number (except 0).

## Example

Let's multiply Row 1 by $\frac{1}{3}$.

$$\left[\begin{array}{ccc|c} 3 & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} ? & * & * & * \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

Why can we multiply Row 1 by $\frac{1}{3}$?

## Notes

1. Write $\frac{1}{3}R_1$ above the arrow to show you multiplied Row 1 by $\frac{1}{3}$.
2. Why can't we multiply by 0?

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$$3x + 6y + 12z = 9 \xrightarrow{\text{Mult. by } \frac{1}{3}} x + 2y + 4z = 3$$

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## Notes

1. Write $\frac{1}{3}R_1$ above the arrow to show you multiplied Row 1 by $\frac{1}{3}$.
2. Multiplying by 0 would eliminate an equation.

# Adding a Multiple of One Row to Another





We can pick any rows and any number.

## Example

Let's multiply Row 1 by -2, then add the result to Row 2, changing only the row to which we add.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ ? & * & * & * \\ 0 & 1 & 4 & 7 \end{array}\right]$$

What does this process look like if we convert back to equations?

## Tip

To mentally compute the new Row 2, calculate one entry at a time. If necessary, write $\begin{bmatrix} -2 & -4 & -8 & -6 \end{bmatrix}$ (this is $-2R_1$) above the original matrix.

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# Interchanging Two Rows





We can interchange any two rows, even if they aren't next to each other.

## Example

Let's switch Rows 2 and 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & -12 & 0 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

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## Why Can We Interchange Any Two Rows?

Since the new system of equations has the same equations as the previous step does, only in a different order, it has the same solution (or solutions). Don't fret: the other two operations don't change the solution either!

# Outline

# What is a "reduced" matrix?

A *reduced matrix*, also known as a *reduced row echelon matrix*, is a matrix that meets the four conditions named below. We'll soon describe each condition separately and clarify with examples.

## Conditions for a Reduced Matrix

- Leading One
- Staircase
- Clear Column
- Zero Row

Note that these names are informal and are intended mainly as memory aids.

Some textbooks combine two of these conditions, so don't be alarmed to see a different number of conditions listed in other sources.

# Leading One



## The Leading One Condition

Moving from left to right, the first nonzero entry in each row (if there is one) is a 1; such 1's are called *leading ones*.

## Which Matrices Below Satisfy the Leading One Condition?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a)

?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(b)

?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

(c)

?

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$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

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(a) (b) (c)

Yes ? ?

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$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a)

Yes

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(b)

Yes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

(c)

?

# Leading One





## The Leading One Condition

Moving from left to right, the first nonzero entry in each row (if there is one) is a 1; such 1's are called *leading ones*.

## Which Matrices Below Satisfy the Leading One Condition?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a)

Yes

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(b)

Yes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

(c)

No

## The Staircase Condition

Each leading one is to the right (not necessarily immediately to the right) of any leading one in a higher row.

## Which Matrices Below Satisfy the Staircase Condition?

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

(a)

?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(b)

?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

(c)

?

## The Staircase Condition

Each leading one is to the right (not necessarily immediately to the right) of any leading one in a higher row.

## Which Matrices Below Satisfy the Staircase Condition?

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

(a)

?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(b)

?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

(c)

?

# Staircase



## The Staircase Condition

Each leading one is to the right (not necessarily immediately to the right) of any leading one in a higher row.

## Which Matrices Below Satisfy the Staircase Condition?

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

(a) No

(b) ?

(c) ?

# Staircase

## The Staircase Condition

Each leading one is to the right (not necessarily immediately to the right) of any leading one in a higher row.

## Which Matrices Below Satisfy the Staircase Condition?

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

(a) No

(b) Yes

(c) ?

# Staircase

## The Staircase Condition

Each leading one is to the right (not necessarily immediately to the right) of any leading one in a higher row.

## Which Matrices Below Satisfy the Staircase Condition?

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

(a)

No

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(b)

Yes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

(c)

No

# Clear Column



## The Clear Column Condition

Each entry above and below a leading one is zero.

## Which Matrices Below Satisfy the Clear Column Condition?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

(a) (b) Why?

? ? ?

Remark: The Staircase Condition ensures that entries *below* a leading 1 are 0, so some texts only emphasize that entries *above* a leading 1 are 0. Either way, a reduced matrix has zeros above and below each leading 1.

# Clear Column

## The Clear Column Condition

Each entry above and below a leading one is zero.

## Which Matrices Below Satisfy the Clear Column Condition?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \qquad \text{Why?}$$

(a) (b)

? ? ?

Remark: The Staircase Condition ensures that entries *below* a leading 1 are 0, so some texts only emphasize that entries *above* a leading 1 are 0. Either way, a reduced matrix has zeros above and below each leading 1.

## The Clear Column Condition

Each entry above and below a leading one is zero.

## Which Matrices Below Satisfy the Clear Column Condition?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \qquad \text{Why?}$$

(a) (b)

No ? ?

Remark: The Staircase Condition ensures that entries *below* a leading 1 are 0, so some texts only emphasize that entries *above* a leading 1 are 0. Either way, a reduced matrix has zeros above and below each leading 1.

# Clear Column





## The Clear Column Condition

Each entry above and below a leading one is zero.

## Which Matrices Below Satisfy the Clear Column Condition?

| (a) | (b) | |
|---|---|---|
| $\left[\begin{array}{cc\|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$ | $\left[\begin{array}{cc\|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$ | Why? |
| No | Yes | ? |

Remark: The Staircase Condition ensures that entries *below* a leading 1 are 0, so some texts only emphasize that entries *above* a leading 1 are 0. Either way, a reduced matrix has zeros above and below each leading 1.

# Clear Column





## The Clear Column Condition

Each entry above and below a leading one is zero.

## Which Matrices Below Satisfy the Clear Column Condition?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

(a)

No

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

(b)

Yes

Why?

In (b), the 2 is above a 1, but not a *leading* 1.

Remark: The Staircase Condition ensures that entries *below* a leading 1 are 0, so some texts only emphasize that entries *above* a leading 1 are 0. Either way, a reduced matrix has zeros above and below each leading 1.

# Clear Column





## The Clear Column Condition

Each entry above and below a leading one is zero.

## Which Matrices Below Satisfy the Clear Column Condition?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

(a)

No

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

(b)

Yes

Why?

In (b), the 2 is above a 1, but not a *leading* 1.

Remark: The Staircase Condition ensures that entries *below* a leading 1 are 0, so some texts only emphasize that entries *above* a leading 1 are 0. Either way, a reduced matrix has zeros above and below each leading 1.

# Zero Row



## The Zero Row Condition

If there are any zero rows (rows consisting entirely of zeros), then they're at the bottom.

## Which Matrices Below Satisfy the Zero Row Condition?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a)

?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b)

?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(c)

?

# Zero Row

Row Reduction in 60 Minutes

G.W. Stanton




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## The Zero Row Condition

If there are any zero rows (rows consisting entirely of zeros), then they're at the bottom.

## Which Matrices Below Satisfy the Zero Row Condition?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a)

?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b)

?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(c)

?

# Zero Row





## The Zero Row Condition

If there are any zero rows (rows consisting entirely of zeros), then they're at the bottom.

## Which Matrices Below Satisfy the Zero Row Condition?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a)

Yes

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b)

?

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(c)

?

# Zero Row



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## The Zero Row Condition

If there are any zero rows (rows consisting entirely of zeros), then they're at the bottom.

## Which Matrices Below Satisfy the Zero Row Condition?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a)

Yes

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b)

No

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(c)

?

# Zero Row

## The Zero Row Condition

If there are any zero rows (rows consisting entirely of zeros), then they're at the bottom.

## Which Matrices Below Satisfy the Zero Row Condition?

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a)

Yes

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b)

No

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(c)

Yes

# Quiz

## Which of These Matrices Are Reduced?

### Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right]$$

Reduced?

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right]$$

Reduced?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Reduced?

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

# Quiz
## Which of These Matrices Are Reduced?



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### Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes | Reduced? | Reduced?

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

# Quiz

## Which of These Matrices Are Reduced?





### Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes — No — Reduced?

1. Leading One ?
2. Staircase *
3. Clear Column *
4. Zero Row *

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

# Quiz

## Which of These Matrices Are Reduced?

### Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes No Reduced?

1. Leading One Yes
2. Staircase ?
3. Clear Column *
4. Zero Row *

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

# Quiz

## Which of These Matrices Are Reduced?

### Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes No Reduced?

1. Leading One Yes
2. Staircase Yes
3. Clear Column ?
4. Zero Row *

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

# Quiz

## Which of These Matrices Are Reduced?



### Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes — No — Reduced?

1. Leading One Yes
2. Staircase Yes
3. Clear Column No
4. Zero Row ?

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

# Quiz
## Which of These Matrices Are Reduced?



### Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes

No

1. Leading One Yes
2. Staircase Yes
3. Clear Column No
4. Zero Row Yes

Reduced?

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

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# Quiz

## Which of These Matrices Are Reduced?

### Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes — No — Yes

1. Leading One Yes
2. Staircase Yes
3. Clear Column No
4. Zero Row Yes

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

# Quiz

Which of These Matrices Are Reduced?



## Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right]$$

Yes

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & .5 \end{array}\right]$$

No

1. Leading One Yes
2. Staircase Yes
3. Clear Column No
4. Zero Row Yes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes

After learning how to use the row operations to reduce a matrix, it will be easier to decide if a matrix is reduced; we'll have another quiz in case you're not convinced.

# Outline

# Solving a System

It's time to solve our first system! Since the beginning of the notes, we've been working with the system below.

$$\begin{aligned} 3x + 6y + 12z &= 9 \\ 2x + 4y - 4z &= 6 \\ y + 4z &= 7 \end{aligned}$$

Remember that to solve it, we first form its augmented matrix.

$$\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

Next, we reduce this matrix.

# Solving a System



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Remember that to solve it, we first form its augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

Next, we reduce this matrix.

# Brief Guidelines for Reducing a Matrix







To avoid common pitfalls, we need guidelines for reducing.

## Brief Guidelines

1. Create leading 1 in first row.
2. Clear new leading 1's column. (make other entries 0).
3. Create leading 1 in second row.
4. Clear new leading 1's column.
5. Repeat.

We actually did the first few steps earlier (shown at right), but the guidelines will show us *why* we used each operation.

$$\left[\begin{array}{ccc|c} 3 & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

$$\downarrow \tfrac{1}{3}R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

$$\downarrow -2R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & -12 & 0 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

$$\downarrow R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right]$$

# Creating a Leading 1



The guidelines tell us to begin by creating a leading 1 in the first row. We often create a row's leading 1 by multiplying by the reciprocal of its leading nonzero entry.

## Example

The leading entry in Row 1 is 3. The reciprocal of 3 is $\frac{1}{3}$.

$$\left[\begin{array}{ccc|c} \boxed{3} & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} ? & * & * & * \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

# Creating a Leading 1



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## Example

The leading entry in Row 1 is 3. The reciprocal of 3 is $\frac{1}{3}$.

$$\left[\begin{array}{rrr|r} \boxed{3} & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{rrr|r} 1 & ? & * & * \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

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# Creating a Leading 1

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$$\left[\begin{array}{ccc|c} \boxed{3} & 6 & 12 & 9 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 2 & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

# Clearing a Column



The guidelines tell us that we should now clear the leading 1's column. To do this, multiply by the opposite of the entry you're trying to clear, then add.

## Example

We have a leading 1 in the first row. Let's use it to clear its column. To clear the 2 below the leading 1, we use the *opposite* of 2, which is -2.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 4 & 3 \\ \boxed{2} & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ ? & * & * & * \\ 0 & 1 & 4 & 7 \end{array}\right]$$

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## Example

We have a leading 1 in the first row. Let's use it to clear its column. To clear the 2 below the leading 1, we use the *opposite* of 2, which is -2.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 4 & 3 \\ \boxed{2} & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & -12 & ? \\ 0 & 1 & 4 & 7 \end{array}\right]$$

# Clearing a Column



The guidelines tell us that we should now clear the leading 1's column. To do this, multiply by the opposite of the entry you're trying to clear, then add.

## Example

We have a leading 1 in the first row. Let's use it to clear its column. To clear the 2 below the leading 1, we use the *opposite* of 2, which is -2.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 4 & 3 \\ \boxed{2} & 4 & -4 & 6 \\ 0 & 1 & 4 & 7 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & -12 & 0 \\ 0 & 1 & 4 & 7 \end{array}\right]$$

# Another Way to Create a Leading 1



In the last step, we cleared the first leading 1's column. Now it's time to create a leading 1 in the second row. To get a leading 1 that will meet the ▸ Staircase Condition, we sometimes need to interchange two rows.

## Example

Since the leading nonzero entry in the second row, -12, is to the right of the leading nonzero entry in the third row, 1, we interchange Rows 3 and 2.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 0 & \boxed{-12} & 0 \\ 0 & \boxed{1} & 4 & 7 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right]$$

## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{?}$$

## Tip

When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} * & * & * & * \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{array}\right]$$

## Tip

When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} ? & * & * & * \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right]$$

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$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & ? & * & * \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right]$$

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When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & ? & * \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right]$$

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When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

## Completing the Reduction

$$\left[\begin{array}{ccr|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccr|c} 1 & 0 & -4 & ? \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right]$$

## Tip

When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

# Quiz

## Can You Finish Reducing the Matrix?



### Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{?}$$

### Tip

When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-\frac{1}{12}R_3}$$

$$\left[\begin{array}{ccc|c} ? & ? & ? & ? \\ ? & ? & ? & ? \\ * & * & * & * \end{array}\right]$$

## Tip

When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-\frac{1}{12}R_3}$$

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## Tip

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### Completing the Reduction

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$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{?}$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-4R_3+R_2} \left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ ? & ? & ? & ? \end{array}\right]$$

### Tip

When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

# Quiz

## Can You Finish Reducing the Matrix?



### Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-\frac{1}{12}R_3}$$

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When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-\frac{1}{12}R_3}$$

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When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

# Quiz

Can You Finish Reducing the Matrix?

## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-\frac{1}{12}R_3}$$

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When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

# Quiz
Can You Finish Reducing the Matrix?

## Completing the Reduction

$$\left[\begin{array}{rrr|r} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{rrr|r} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-\frac{1}{12}R_3}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-4R_3+R_2} \left[\begin{array}{rrr|r} * & * & * & * \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & 0 \end{array}\right]$$

## Tip

When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

Row Reduction in 60 Minutes

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## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-\frac{1}{12}R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{-4R_3+R_2}{\overset{?}{\longrightarrow}} \left[\begin{array}{ccc|c} * & * & * & * \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

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When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

# Quiz

## Can You Finish Reducing the Matrix?

### Completing the Reduction

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# Quiz

Can You Finish Reducing the Matrix?



## Completing the Reduction

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## Completing the Reduction

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## Completing the Reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -11 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -12 & 0 \end{array}\right] \xrightarrow{-\frac{1}{12}R_3}$$

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## Tip

When using a leading 1 to clear its column, you can perform more than one operation per step to save time.

# Converting Back to a System of Equations

## Obtaining the Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \begin{aligned} x &= -11 \\ y &= 7 \\ z &= 0 \end{aligned}$$

We can also write the solution as $(-11, 7, 0)$.

We've solved our first system! To be safe, let's check our answer: plug it into each equation in the original system. If any equation is false (such as 3=5), then $(-11, 7, 0)$ is not actually a solution.

## Checking the solution

$$\begin{aligned} 3x + 6y + 12z &= 9 \\ 2x + 4y - 4z &= 6 \\ y + 4z &= 7 \end{aligned} \longrightarrow \begin{aligned} 3(-11) + 6(7) + 12(0) &= 9 \\ 2(-11) + 4(7) - 4(0) &= 6 \\ (7) + 4(0) &= 7 \end{aligned}$$

# Converting Back to a System of Equations





## Obtaining the Solution

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# Guidelines for Reducing a Matrix, Revisited

We can avoid trouble by sticking to the following guidelines.

## Guidelines

1. Beginning with first row . . .
   (a) ?

Remark: it's possible that two people applying these guidelines to the same problem will not use the same sequence of row operations. However, they will always get the same reduced matrix.

# Guidelines for Reducing a Matrix, Revisited





We can avoid trouble by sticking to the following guidelines.

## Guidelines

1. Beginning with first row . . .
   (a) Create leading 1: Which operation(s) do we want to use?

Remark: it's possible that two people applying these guidelines to the same problem will not use the same sequence of row operations. However, they will always get the same reduced matrix.

# Guidelines for Reducing a Matrix, Revisited





We can avoid trouble by sticking to the following guidelines.

## Guidelines

1. Beginning with first row ...
   (a) Create leading 1: multiply by reciprocal, or interchange rows.
   (b) What's next?

Remark: it's possible that two people applying these guidelines to the same problem will not use the same sequence of row operations. However, they will always get the same reduced matrix.

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Row Reduction in 60 Minutes

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We can avoid trouble by sticking to the following guidelines.

## Guidelines

1. Beginning with first row . . .
   (a) Create leading 1: multiply by reciprocal, or interchange rows.
   (b) Clear its column: Which operation(s) do we want to use?

Remark: it's possible that two people applying these guidelines to the same problem will not use the same sequence of row operations. However, they will always get the same reduced matrix.

# Guidelines for Reducing a Matrix, Revisited

We can avoid trouble by sticking to the following guidelines.

## Guidelines

1. Beginning with first row . . .
   (a) Create leading 1: multiply by reciprocal, or interchange rows.
   (b) Clear its column: multiply by opposite and add.
2. Then what?

Remark: it's possible that two people applying these guidelines to the same problem will not use the same sequence of row operations. However, they will always get the same reduced matrix.

# Guidelines for Reducing a Matrix, Revisited





We can avoid trouble by sticking to the following guidelines.

## Guidelines

1. Beginning with first row . . .
   (a) Create leading 1: multiply by reciprocal, or interchange rows.
   (b) Clear its column: multiply by opposite and add.
2. Create leading 1 in second row, clear its column.
3. And then?

Remark: it's possible that two people applying these guidelines to the same problem will not use the same sequence of row operations. However, they will always get the same reduced matrix.

# Guidelines for Reducing a Matrix, Revisited





We can avoid trouble by sticking to the following guidelines.

## Guidelines

1. Beginning with first row ...
   (a) Create leading 1: multiply by reciprocal, or interchange rows.
   (b) Clear its column: multiply by opposite and add.
2. Create leading 1 in second row, clear its column.
3. Repeat with each row (except for zero rows).

Remark: it's possible that two people applying these guidelines to the same problem will not use the same sequence of row operations. However, they will always get the same reduced matrix.

# Guidelines for Reducing a Matrix, Revisited



We can avoid trouble by sticking to the following guidelines.

## Guidelines

1. Beginning with first row . . .
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   (b) Clear its column: multiply by opposite and add.
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3. Repeat with each row (except for zero rows).

Remark: it's possible that two people applying these guidelines to the same problem will not use the same sequence of row operations. However, they will always get the same reduced matrix.

# Two Ways to Decide When a Matrix Is Reduced





In a previous quiz, we decided if a matrix was reduced by checking the four required conditions. We're ready for another approach that often takes less effort.

## How to Decide When a Matrix Is Reduced

- Check each of the four required conditions, or
- Attempt to reduce the matrix according to our guidelines.
  - If there is nothing to do, the matrix is already reduced!
  - If there is work to do, the matrix is not reduced!

Next, we'll take our second quiz, as promised.

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Next, we'll take our second quiz, as promised.

## Example

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Reduced? Reduced? Reduced?

# Another Quiz

## Which of These Matrices Are Reduced?

### Example

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No, must clear fourth column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Reduced?

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Reduced?

# Another Quiz

## Which of These Matrices Are Reduced?

### Example

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

No, must clear fourth column.

Yes

Reduced?

# Another Quiz

## Which of These Matrices Are Reduced?



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### Example

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No, must clear fourth column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Yes

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

No, must switch Rows 2 and 3.

# Outline

# How Many Solutions Are Possible?

## The Graphical Perspective

Earlier, we solved a system of equations and found exactly one solution: $(-11, 7, 0)$. However, not every system of linear equations has exactly one solution.

Can a system of linear equations have exactly two solutions? The answer turns out to be "no." How many solutions can a system of linear equations have? We'll investigate the different possibilities next.

To begin, it will help us to look at solutions from a graphical perspective.

# Does the Point Lie on the Graph?

Does the point $(1, 4)$ lie on the graph of $2x + 3y = 6$? Since $2(1) + 3(4) \neq 6$, $(1, 4)$ does not lie on the graph. If plugging in a point's coordinates results in a true equation, then the point does lie on the graph.

Does $(4, 3)$ lie on the graph of $7x + 5y = 35$?

In summary, the graph of an equation consists of those points whose coordinates make the equation true.

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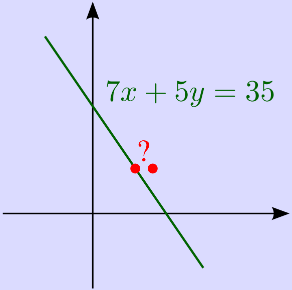
**Does $(4, 3)$ lie on the graph of $7x + 5y = 35$?**

What do you think?

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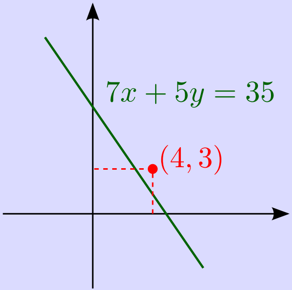


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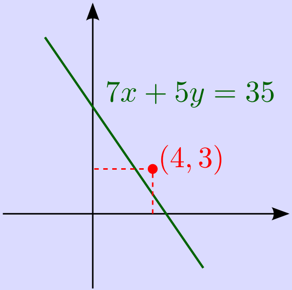


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# Solutions and Graphs





We called $(-11, 7, 0)$ a solution to our first system of equations, because when we substituted 11, 7, and 0 for $x$, $y$, and $z$, each equation was made true (was *satisfied*).

Similarly, $(3, 6)$ is a solution to the system

$$x + y = 9$$
$$2x - y = 0$$

because this point satisfies each equation.

In other words, a solution is a point that lies on the graph of each equation.

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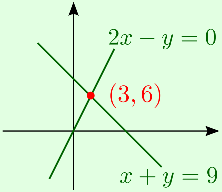

# The Three Cases
## (Graphical Perspective)

Row Reduction in 60 Minutes

G.W. Stanton






How many solutions can a linear system have? Let's say we're given equations for two lines. The number of solutions is the number of points where the lines intersect.

Try drawing graphs that show three different ways the lines can intersect (or fail to intersect).

| | | |
|---|---|---|
| ? | ? | ? |

Although these pictures only represent the two-variable case ($x$ and $y$), a linear system will always have no solution at all, exactly one solution, or infinitely many solutions, even with more than two variables.

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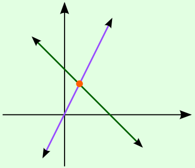

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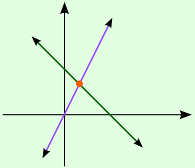

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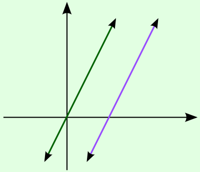

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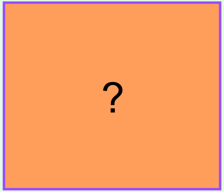


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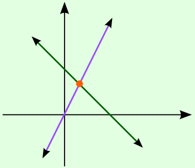

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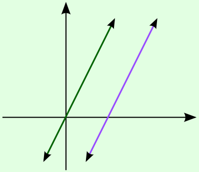

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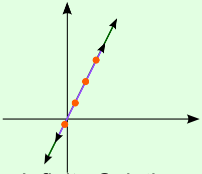

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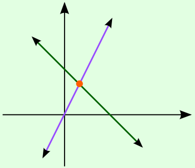
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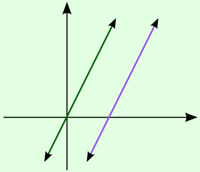
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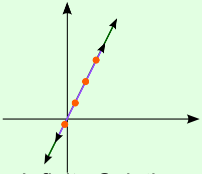
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# The Unique Solution
(Algebraic Perspective)



How can you determine the number of solutions from the augmented matrix, once it's reduced? If there is a unique solution, it will be right in front of us, just like when we solved our first system.

**Example: Unique Solution**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \longrightarrow$$

We have $x = 7$, $y = -2$, and $z = 5$, so there is one solution (7,-2,5). If it's not clear why this is one solution and not three, . . .

Hmmm. Can you think of a way to make sense of this?

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...it helps to think of (7,-2,5) as a single point [in three dimensions], just as an ordered pair $(x, y)$ represents a single point in the $xy$-plane.

# No Solution - Part 1
(Algebraic Perspective)



How will we know if a system of linear equations has no solution? As always, we begin by reducing the augmented matrix and converting back to equations.

## Example: No Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow$$

The equations $x = 7$ and $y = -2$ correspond to the first two rows. What equation corresponds to the third row?

How can we make sense of this result?

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We know $0 \cdot (\text{any number}) = 0$, so $0x = 0$, $0y = 0$, and $0z = 0$. Therefore, $0x + 0y + 0z$ is just 0, which means the third row tells us $0 = 1$!

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How can we make sense of this result?

Since there are no values for $x$, $y$, and $z$ that make $0x + 0y + 0z = 1$ true, the other two equations don't matter: it's impossible for all three equations to be satisfied, so the system has no solution.

# No Solution - Part 2
(Algebraic Perspective)





Whenever there is no solution, there will be the equation $0 = 1 \ldots$

…basically. If the system has no solution, then as long as the augmented matrix has been *reduced*, it will *always* contain a row corresponding to $0 = 1$.

In practice, however, it's not always necessary to fully reduce the augmented matrix when a system has no solution. For example, if the augmented matrix contains a row corresponding to 0=5, we can conclude that the given system has no solution (without continuing to reduce), because there are no values of $x$, $y$, and $z$ that satisfy this equation.

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# Infinitely Many Solutions - Part 1
(Algebraic Perspective)

How will we know if a system of linear equations has infinitely many solutions?

## Example: Infinitely Many Solutions

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow$$

Is this matrix reduced? Using $r$, $s$, $t$, $u$, and $v$, what equations correspond to this matrix?

How can we be sure there are infinitely many solutions?

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[Yes, the matrix is reduced.]

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[Yes, the matrix is reduced.]

How can we be sure there are infinitely many solutions?

There must be at least one solution, because we don't see $0 = 1$. We don't immediately see a unique solution; this tells us the solution isn't unique. There must be infinitely many solutions.

Great! We know there are infinitely many solutions. But wait, there's more! Just one piece of the puzzle remains.

## Determining the General Solution

1. *Identify the columns* (to the left of the rightmost column) *that do not contain a leading one.*

   We will use the word *parameter* to refer to each of the variables corresponding to these columns.

2. *Move the parameters to the other side* of each equation.

Next, we'll apply these steps to complete the example from the previous slide.

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## Example: Infinitely Many Solutions (Continued)

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Which columns do not contain a leading one?

What do we get when we convert back to equations?

Although there are infinitely many solutions, we cannot always obtain a solution by arbitrarily choosing numbers for each variable. We CAN arbitrarily choose numers for the PARAMETERS $s$ and $u$.

For example, if we choose $s = 1$ and $u = 2$, our equations tell us $r = 2 - (1) - 4(2) = -7$ and $t = 1 - 3(2) = -5$, and we obtain a solution $(-7, 1, -5, 2, 0)$. Different values for $s$ and $u$ produce different solutions.

## Example: Infinitely Many Solutions (Continued)

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What do we get when we move the parameters over?

[The parameters are highlighted.]

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# Infinitely Many Solutions - Part 3
(Algebraic Perspective)

## Example: Infinitely Many Solutions (Continued)

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When you need to carefully express your answer, you have a variety of options. Different teachers will tend to follow different conventions, but a few options are given below, using the previous example.

- $r = 2 - s - 4u$, $t = 1 - 3u$, $v = 0$; $s$ and $u$ are arbitrary
- $r = 2 - s - 4u$, $s = s$, $t = 1 - 3u$, $u = u$, and $v = 0$
- $\{(2 - s - 4u, s, 1 - 3u, u, 0) : s, u \in \mathbb{R}\}$

In the second option, the equations $s = s$ and $u = u$ indicate that $s$ and $u$ may be specified arbitrarily. The third option makes use of something called set-builder notation (you can ignore this option if your course doesn't use this notation), and it indicates that we are working only with what are known as *real numbers* (these notes and most introductory courses work only with real numbers).

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# Infinitely Many Solutions - Part 4
(Algebraic Perspective)



When you need to carefully express your answer, you have a variety of options. Different teachers will tend to follow different conventions, but a few options are given below, using the previous example.

- $r = 2 - s - 4u$, $t = 1 - 3u$, $v = 0$; $s$ and $u$ are arbitrary
- $r = 2 - s - 4u$, $s = s$, $t = 1 - 3u$, $u = u$, and $v = 0$
- $\{(2 - s - 4u, s, 1 - 3u, u, 0) : s, u \in \mathbb{R}\}$

In the second option, the equations $s = s$ and $u = u$ indicate that $s$ and $u$ may be specified arbitrarily. The third option makes use of something called set-builder notation (you can ignore this option if your course doesn't use this notation), and it indicates that we are working only with what are known as *real numbers* (these notes and most introductory courses work only with real numbers).

# What We've Accomplished



We now know how to solve any system of linear equations in any number of variables!

Woo hoo!

All that remains on the road to mastery is practice.

# Feedback



Thanks for taking the time to work through these notes!

## Comments and Suggestions

If you ...

- found these notes helpful,
- found a typo, or
- have a suggestion,

please let Higher Math Help know.